Three topics in the Schwinger model*

Ph. de Forcrand,[a] J.E. Hetrick,[b] T. Takaishi [c] and A.J. van der Sijs [a]

[a]Swiss Center for Scientific Computing, ETH-Zürich, ETH-Zentrum, CH-8092 Zürich, Switzerland

[b]Physics Department, University of the Pacific, Stockton, CA 95211, USA

[c]Hiroshima University of Economics, Hiroshima, Japan 731-01

1. We compare Monte Carlo results with analytic predictions for the fermion condensate, in the massive one-flavour Schwinger model. 2. We illustrate on the Schwinger model how to facilitate transitions between topological sectors by a simple reweighting method. 3. We discuss exact, non-perturbative improvement of the gauge sector.

1. FERMION CONDENSATE

In the massless (one-flavour) Schwinger model, the fermion condensate is

$$\langle -\bar{\psi}\psi \rangle / \mu = e^{\gamma}/2\pi = 0.283466, \qquad (1)$$

in units of $\mu = g/\sqrt{\pi}$. For non-zero fermion mass m, no exact expression is known, but there are some results from analytical approximations and lattice simulations [1]. No clear agreement has been reached, though.

Here we report on a simulation using the R-algorithm [2] with staggered fermions, in which extrapolations to zero step-size, to zero lattice spacing and to infinite volume are performed. We focus on two values of m/μ: 0.33 and 1.00.

Fig. 1 shows the approach to the continuum limit for a given physical volume: the condensate is plotted against $1/\beta$ for a series of simulations on $(L/a) \times (2L/a)$ lattices ($L/a = 6, 8, \ldots, 16$), keeping $am\sqrt{\beta\pi} = 0.33$ and $(L/a)/\sqrt{\beta} = 4\sqrt{2}$ fixed. The condensate for the free theory is subtracted (in two different ways, see caption) in order to cancel the UV divergence present in the massive case. A linear extrapolation to the continuum limit, accounting for $\mathcal{O}(1/\beta) \sim \mathcal{O}(a^2)$ perturbative scaling violations, works well and is seen to be very important, given the large effect. After additional extrapolations to infinite volume and zero MD step-size, we find for the
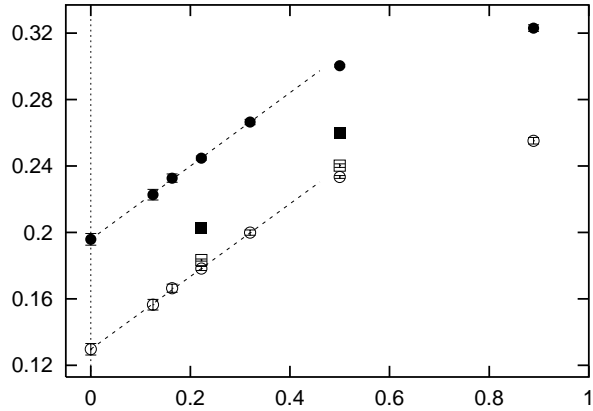

Figure 1. Fermion condensate against $1/\beta$, with extrapolation, for fixed physical volume and fermion mass. The two sets of points correspond to subtraction of the free condensate in the same finite volume (solid circles) or in infinite volume (open circles). The four squares correspond to 2×2 times as large volumes.

(subtracted) condensate:

$$\langle -\bar{\psi}\psi \rangle_{\mathrm{sub}} / \mu = 0.141(5) \quad (m/\mu = 0.33), \qquad (2)$$

and similarly for the larger mass:

$$\langle -\bar{\psi}\psi \rangle_{\mathrm{sub}} / \mu = 0.084(3) \quad (m/\mu = 1.00). \qquad (3)$$

In fig. 2 these data are compared with a recent analytical result by Hosotani[2] [3], expected

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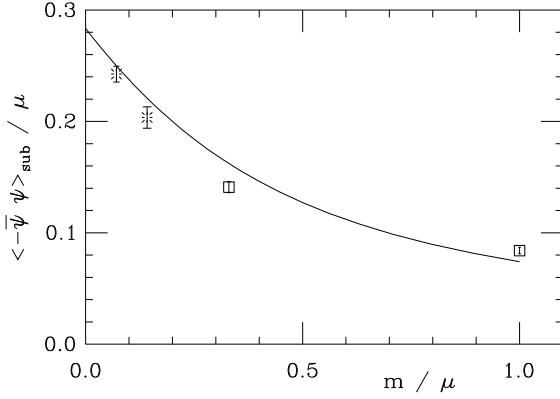[2]We thank Y. Hosotani for discussions on this topic.

Figure 2. The fermion condensate in infinite volume, in the continuum limit (squares) compared with the analytical small-m result of Ref. [3] (solid curve). The two asterisks correspond to our reanalysis of data from Ref. [4].

to hold in the small-m region. For comparison, we have also plotted our reanalysis (*i.e.,* subtraction of the free condensate and subsequent linear extrapolation to $1/\beta = 0$) of data from Ref. [4] for the non-compact model on a 64×64 lattice.

2. SAMPLING TOPOLOGICAL SECTORS AT HIGH β

A correct sampling of the different topological sectors is essential for a correct determination of quantities like the fermionic condensate in the Schwinger model. In the usual importance sampling algorithms, however, transitions between sectors are suppressed by a factor of order $\exp(2\beta)$ in the continuum limit, implying *exponential* critical slowing down for the topological autocorrelation time.

We propose the following reweighting technique to overcome this problem. We add to the action the quantity

$$\Delta S = -\sum_x \alpha \exp\left[-\frac{(\theta_{\mu\nu}(x) - \pi)^2}{2\theta_0^2}\right] \delta_{x,x_0}, \qquad (4)$$

and compensate for this by including a factor $\exp(\beta\Delta S)$ in the observables. Here x_0 is the site
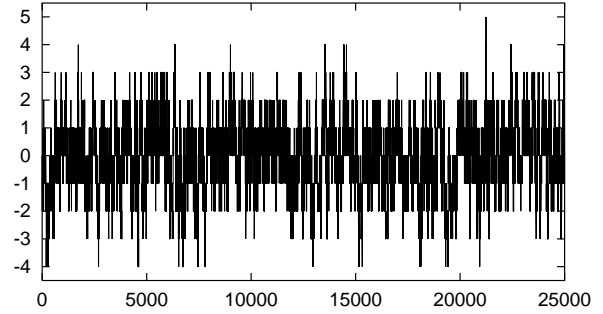

Figure 3. Topological charge history (in units of M.D. time) for a simulation on an 18×36 lattice at $\beta = 10.125$, using reweighting. For comparison, without reweighting we observed only *one* transition in $200\,000$ units of M.D. time.

at which the plaquette angle $\theta_{\mu\nu}$ is closest to π of all the plaquettes in the configuration under consideration. This procedure increases the topological transition rate by strongly enhancing the probability that the plaquette at x_0 goes through π. Note that x_0 may vary from one configuration to the next, but the essential point is that the prescription assigns a unique action to each configuration. It can be shown that this leads to reversible molecular dynamics, ensuring detailed balance. The computational overhead is negligible.

The example in Fig. 3 shows that this method drastically improves sampling of the topological sectors (ergodicity).

The parameters α and θ_0 can be used for optimization. For example, they can be adjusted such that the distribution (histogram) of $\theta_{\mu\nu}(x_0)$ is roughly flat over most of the interval $[-\pi, \pi]$. On large lattices, $\theta_{\mu\nu}(x_0)$ gets closer to π, such that smaller α and θ_0 suffice for the same transition rate, and the bias due to Eq. (4) gets smaller.

This reweighting procedure enhances a local lattice artefact, which acts like a "saddle point" between topological sectors in the lattice formulation; in the continuum theory, these sectors are completely disconnected. This observation may guide a simple generalization to non-abelian gauge theories in 4 dimensions.

3. NON-PERTURBATIVE IMPROVEMENT

Two-dimensional U(1) lattice gauge theory with the standard plaquette action is exactly solvable. In particular, in the infinite-volume limit the expectation value of any rectangular Wilson loop of charge q satisfies an exact area law with string tension

$$a^2 \sigma_q(\beta) = \ln \frac{I_0(\beta)}{I_q(\beta)} \tag{5}$$

$$= q^2 \left(\frac{1}{2\beta} + \frac{1}{4\beta^2} \right) + \mathcal{O}\left(\frac{1}{\beta^3} \right), \tag{6}$$

where $I_{0,q}(\beta)$ are the Bessel functions. For the non-compact action, on the other hand (cf. also the Manton action), one finds $\sigma_q = q^2/2\beta$ ("perturbative scaling") exactly. We use this observation as a guideline for non-perturbative improvement[3] of the compact plaquette action: tune the coefficient of the improvement term such that the string tension equals $q^2/2\beta$.

We consider the addition of an adjoint (squared plaquette) term to the standard action:

$$\Delta S = \sum_p c(\beta) \left[\frac{1 - \cos\theta_p}{3} - \frac{1 - \cos 2\theta_p}{12} \right]. \tag{7}$$

In this case, the string tension is given by Eq. (5) with I_n replaced with λ_n, defined as

$$\lambda_n(\beta_f, \beta_a) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \, \cos(n\theta) \, e^{\beta_f \cos\theta + \beta_a \cos 2\theta}, \tag{8}$$

with $\beta_f = \beta\,(1 + c(\beta)/3)$, $\beta_a = -\beta\,c(\beta)/12$. The λ_n are calculated numerically.

In Fig. 4 we compare various choices for $c(\beta)$. By taking $c \equiv 1$, the θ_p^4 term in the action, and hence the β^{-2} term in $a^2\sigma_q$ (6), are cancelled (perturbative improvement): the slope of the relative "error" in the string tension (fig. 4b) increases from 1 to 2. The error is seen to be further reduced when this coefficient is tadpole-improved, by taking $c(\beta) = 1/\langle \Box(\beta) \rangle$ self-consistently. Non-perturbative improvement amounts to tuning $c(\beta)$ such that $a^2\sigma_q = q^2/2\beta$

[3]This is different from the usual improvement where the expansion of the plaquette angle in terms of the continuum gauge field plays an essential role.
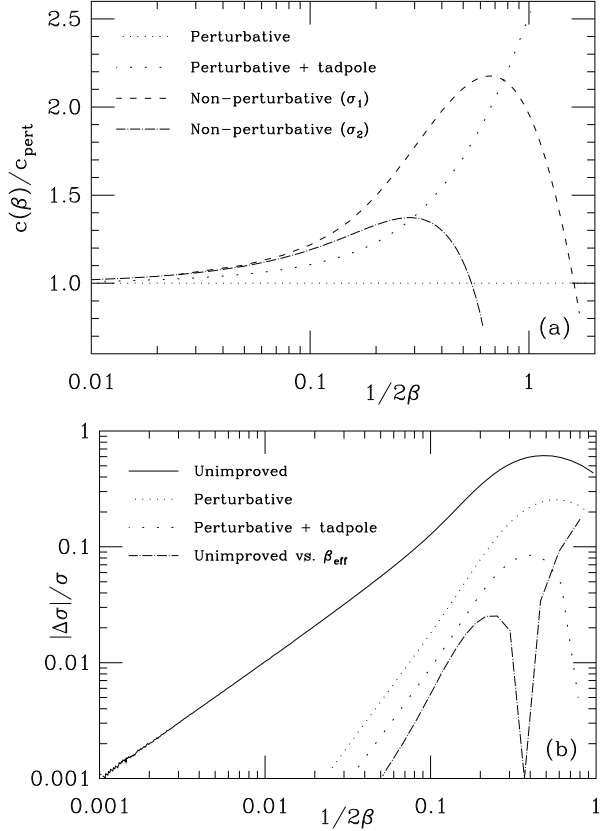

Figure 4. $c(\beta)$ for the improvement schemes described in the text (a) and relative deviation of $a^2\sigma_1$ from the value $1/2\beta$ (b). For the dash-dotted curve in (b), the horizontal axis is $1/2\beta_{\mathrm{eff}}$.

for all β (for a given q). Note that a very good result is obtained by simply using the unimproved $(c \equiv 0)$ Wilson action but expressing the results in terms of the effective coupling $\beta_{\mathrm{eff}} = \beta\,\langle \Box(\beta) \rangle$ suggested by the tadpole scheme.

REFERENCES

1. For a nice early paper, see S.R. Carson and R.D. Kenway, Ann. Phys. 166 (1986) 364.
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3. Y. Hosotani, fig. 2 of hep-th/9703153.
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